The Curl of Conservative Fields

Recall that every **conservative** field can be written as the gradient of some scalar field:

$$\mathbf{C}(\bar{r}) = \nabla g(\bar{r})$$

Consider now the **curl of a conservative field**:

$$\nabla x \mathbf{C}(\bar{r}) = \nabla x \nabla g(\bar{r})$$

Recall that if $\mathbf{C}(\bar{r})$ is expressed using the **Cartesian** coordinate system, the curl of $\mathbf{C}(\bar{r})$ is:

$$\nabla x \mathbf{C}(\bar{r}) = \left[\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right]\hat{a}_x + \left[\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right]\hat{a}_y + \left[\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right]\hat{a}_z$$

Likewise, the **gradient** of $g(\bar{r})$ is:

$$\nabla g(\bar{r}) = \mathbf{C}(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x}\hat{a}_x + \frac{\partial g(\bar{r})}{\partial y}\hat{a}_y + \frac{\partial g(\bar{r})}{\partial z}\hat{a}_z$$

Therefore:

$$C_x(\bar{r}) = \frac{\partial g(\bar{r})}{\partial x}$$
$$C_y(\bar{r}) = \frac{\partial g(\bar{r})}{\partial y}$$
$$C_z(\bar{r}) = \frac{\partial g(\bar{r})}{\partial z}$$

Combining these two results:

$$\begin{aligned}\nabla x \nabla g(\bar{r}) = &\left[\frac{\partial^2 g(\bar{r})}{\partial y \partial z} - \frac{\partial^2 g(\bar{r})}{\partial z \partial y}\right]\hat{a}_x \\ &+ \left[\frac{\partial^2 g(\bar{r})}{\partial z \partial x} - \frac{\partial^2 g(\bar{r})}{\partial x \partial z}\right]\hat{a}_y \\ &+ \left[\frac{\partial^2 g(\bar{r})}{\partial x \partial y} - \frac{\partial^2 g(\bar{r})}{\partial y \partial x}\right]\hat{a}_z\end{aligned}$$

Since, for example:

$$\frac{\partial^2 g(\bar{r})}{\partial y \partial z} = \frac{\partial^2 g(\bar{r})}{\partial z \partial y},$$

each component of $\nabla x \nabla g(\bar{r})$ is then equal to **zero**, and we can say:

$$\nabla x \nabla g(\bar{r}) = \nabla x \mathbf{C}(\bar{r}) = 0$$

The **curl** of every **conservative** field is **equal to zero** !

Likewise, we have determined that:

$$\nabla \mathrm{x} \nabla g(\bar{r}) = 0$$

for **all** scalar functions $g(\bar{r})$.

Q: *Are there some **non-**conservative fields whose curl is also equal to zero?*

A: **NO!** The curl of a conservative field, and **only** a conservative field, is equal to **zero**.

Thus, we have way to **test** whether some vector field $\mathbf{A}(\bar{r})$ is conservative: **evaluate its curl**!

1. If the result **equals zero**—the vector field **is** conservative.

2. If the result is **non-zero**—the vector field **is not** conservative.

Let's again **recap** what we've learned about **conservative** fields:

1. The line integral of a conservative field is **path independent**.

2. Every conservative field can be expressed as the **gradient** of some scalar field.

3. The gradient of **any** and **all** scalar fields is a conservative field.

4. The line integral of a conservative field around any **closed** contour is equal to zero.

5. The **curl** of every conservative field is equal to **zero**.

6. The **curl** of a vector field is zero **only** if it is conservative.